THE MÖBIUS-POMPEÏU METRIC PROPERTY

Branko J. Malešević

University of Belgrade, Faculty of Electrical Engineering,
Bulevar kralja Aleksandra 73, Belgrade, Serbia & Montenegro
E-mail: malesevic@kiklop.etf.bg.ac.yu

In the paper we consider an extension of Möbius-Pompeïu theorem of the elementary geometry over metric spaces. We specially take into consideration Ptolemaic metric spaces.

1 The Möbius-Pompeïu theorem and metric spaces

In this paper we consider the following statement of elementary geometry [1], [2]:

Theorem 1.1 (Möbius, Pompeïu) *Let ABC be an equilateral triangle and M any point in its plane. Then segments MA, MB and MC are sides of a triangle.*

Let us consider analogous problem for the metric space (X, d) with at least four points. Let $A, B, C \in X$ be three fixed points. Then, for the point $M \in X$ we suppose that *a triangle can be formed* from the distances $d_1 = d(M, A)$, $d_2 = d(M, B)$ and $d_3 = d(M, C)$ iff the following conjunction of inequalities is true:

$$(1.1) \qquad d_1 + d_2 - d_3 \geq 0 \quad \text{and} \quad d_2 + d_3 - d_1 \geq 0 \quad \text{and} \quad d_3 + d_1 - d_2 \geq 0.$$

If in conjunction (1.1) at least one equality is true, then we suppose that *a degenerative triangle can be formed*. If in (1.1) sharp inequalities are true:

$$(1.2) \qquad d_1 + d_2 - d_3 > 0 \quad \text{and} \quad d_2 + d_3 - d_1 > 0 \quad \text{and} \quad d_3 + d_1 - d_2 > 0,$$

then we suppose that *a non-degenerative triangle can be formed*. In this case, for the point M, for which the conjunction (1.2) is true, we define that *point have Möbius-Pompeïu metric property*. The main subject of this paper is to determine points M which do not have Möbius-Pompeïu metric property, i.e. these points which fulfill the following disjunction of the inequalities:

$$(1.3) \qquad d_1 + d_2 - d_3 \leq 0 \quad \text{or} \quad d_2 + d_3 - d_1 \leq 0 \quad \text{or} \quad d_3 + d_1 - d_2 \leq 0.$$

Let us notice that the point $M \in \{A, B, C\}$ do not have Möbius-Pompeïu metric property. Thus in consideration which follows, we assume that the metric space (X, d) has at least four points.

2000 Mathematics Subject Classification: 54E35, 51M16.

Research partially supported by the MNTRS, Serbia & Montenegro, Grant No. 1861.

2 Ptolemaic metric spaces

A metric space (X, d) is called *Ptolemaic metric space* if Ptolemaic inequality holds:

$$d(x_1, x_2)d(x_3, x_4) \leq d(x_2, x_4)d(x_1, x_3) + d(x_1, x_4)d(x_2, x_3) \tag{2.1}$$

for every $x_1, x_2, x_3, x_4 \in X$ [3]. A normed space $(X, |\,.\,|)$ is *Ptolemaic normed space* if metric space (X, d) is Ptolemaic with the distance $d(x, y) = |x - y|$. Let us notice that the following lemma is true [3]:

Lemma 2.1 *A normed space is Ptolemaic iff it is an inner product space.*

We give two basic examples of Ptolemaic spaces [3].

Example 2.2 $\mathbf{1^0}$**.** *The space* $\mathbf{R}^n$ *with the Euclidean metric* $\mathtt{d}(x, y) = |x - y|$ *is a Ptolemaic metric space.*
$\mathbf{2^0}$**.** *The space* $\mathbf{R}^n$ *with the chordal metric on the unit Riemann sphere* $\overline{\mathtt{d}}(x, y) = \dfrac{2|x - y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}$ *is a Ptolemaic metric space.*

We will illustrate following considerations with the previous examples of Ptolemaic metric spaces in the case of dimension $n = 2$.

3 The main results

Let (X, d) be a metric space. Let us fix three points $A, B, C \in X$ and form distances:

$$a = d(B, C),\ b = d(C, A),\ c = d(A, B). \tag{3.1}$$

For any point $M \in X$ let us form distances:

$$d_1 = d(M, A),\ d_2 = d(M, B),\ d_3 = d(M, C). \tag{3.2}$$

Inequality $d_2 + d_3 \leq d_1$

Let us determine a set of M points of metric spaces X for which the following inequality is true:

$$d_2 + d_3 \leq d_1. \tag{3.3}$$

Let us form two functions:

$$\alpha_1 = \alpha_1(M) = 4d_2^2 d_3^2 - \big(d_1^2 - (d_2^2 + d_3^2)\big)^2, \tag{3.4}$$

$$\beta_1 = \beta_1(M) = d_2^2 + d_3^2 - d_1^2. \tag{3.5}$$

Lemma 3.1 *For points A, B and C inequality $\alpha_1 \leq 0$ is true.*

Proof. For point A: $d_1 = 0$ and $\alpha_1 = -(c^2 - b^2)^2 \leq 0$ are true. Similarly, the previous inequality is true for the points B and C. ■

Example 3.2 *Let vertices A, B, C of the triangle ABC in the plane $\mathbf{R}^2$ be given by coordinates $A(a_1, b_1)$, $B(a_2, b_2)$, $C(a_3, b_3)$ and let $M(x, y)$ be any point in its plane.*

$\mathbf{1^0}$. *Let us in the plane $\mathbf{R}^2$ use Euclidean metric $\mathtt{d}$. Let us specify the form of term α_1 and β_1 which correspond to functions* (3.4) *and* (3.5) *respectively. It is true*:

$$(3.6)\quad \alpha_1 = \mathtt{k}(x^2+y^2)^2+(\mathtt{A}_1x+\mathtt{B}_1y)(x^2+y^2)+\mathtt{C}_1x^2+\mathtt{D}_1xy+\mathtt{E}_1y^2+\mathtt{F}_1x+\mathtt{G}_1y+\mathtt{H}_1,$$

for some coefficients $\mathtt{k}, \mathtt{A}_1, \mathtt{B}_1, \mathtt{C}_1, \mathtt{D}_1, \mathtt{E}_1, \mathtt{F}_1, \mathtt{G}_1, \mathtt{H}_1 \in \mathbf{R}$ $(\mathtt{k}=3)$. Equality $\alpha_1 = 0$ determines the algebraic curve of the fourth order. By inequality $\alpha_1 < 0$ we determine the interior of the previous curve. Also, it is true:

$$(3.7)\qquad \beta_1 = \mathtt{A}_2(x^2+y^2)+\mathtt{B}_2x+\mathtt{C}_2y+\mathtt{D}_2,$$

for some coefficients $\mathtt{A}_2, \mathtt{B}_2, \mathtt{C}_2, \mathtt{D}_2 \in \mathbf{R}$ $(\mathtt{A}_2 = 1)$. If $\mathtt{B}_2^2+\mathtt{C}_2^2 > 4\mathtt{D}_2$ equality $\beta_1 = 0$ is possible and determines the circle. Then by inequality $\beta_1 < 0$ we determine the interior of the circle.

$\mathbf{2^0}$. *Let us in the plane $\mathbf{R}^2$ use chordal metric $\overline{\mathtt{d}}$. Let us specify the form of the term $\overline{\alpha}_1$ and $\overline{\beta}_1$ which correspond to functions* (3.4) *and* (3.5) *respectively. It is true*:

$$(3.8)\quad \overline{\alpha}_1 = \frac{\overline{\mathtt{k}}(x^2+y^2)^2+(\overline{\mathtt{A}}_1x+\overline{\mathtt{B}}_1y)(x^2+y^2)+\overline{\mathtt{C}}_1x^2+\overline{\mathtt{D}}_1xy+\overline{\mathtt{E}}_1y^2+\overline{\mathtt{F}}_1x+\overline{\mathtt{G}}_1y+\overline{\mathtt{H}}_1}{(1+x^2+y^2)^2(1+a_1^2+b_1^2)^2(1+a_2^2+b_2^2)^2(1+a_3^2+b_3^2)^2},$$

for some coefficients $\overline{\mathtt{k}}, \overline{\mathtt{A}}_1, \overline{\mathtt{B}}_1, \overline{\mathtt{C}}_1, \overline{\mathtt{D}}_1, \overline{\mathtt{E}}_1, \overline{\mathtt{F}}_1, \overline{\mathtt{G}}_1, \overline{\mathtt{H}}_1 \in \mathbf{R}$. If $\overline{\mathtt{k}} \neq 0$ equality $\overline{\alpha}_1 = 0$ determines the algebraic curve of the fourth order. Then by inequality $\overline{\alpha}_1 < 0$ we determine the interior of the previous curve. Also, it is true:

$$(3.9)\qquad \overline{\beta}_1 = \frac{\overline{\mathtt{A}}_2(x^2+y^2)+\overline{\mathtt{B}}_2x+\overline{\mathtt{C}}_2y+\overline{D}_2}{(1+x^2+y^2)(1+a_1^2+b_1^2)(1+a_2^2+b_2^2)(1+a_3^2+b_3^2)},$$

for some coefficients $\overline{\mathtt{A}}_2, \overline{\mathtt{B}}_2, \overline{\mathtt{C}}_2, \overline{\mathtt{D}}_2 \in \mathbf{R}$. If $\overline{\mathtt{A}}_2 \neq 0$ and $\overline{\mathtt{B}}_2^2+\overline{\mathtt{C}}_2^2 > 4\overline{\mathtt{A}}_2\overline{\mathtt{D}}_2$ equality $\overline{\beta}_1 = 0$ is possible and determines the circle. Then by the inequality $\overline{\beta}_1 < 0$ we determine the interior of the circle.

Further, let us notice that for the function α_1:

$$(3.10)\qquad \alpha_1 = (d_2+d_3-d_1)(d_3+d_1-d_2)(d_1+d_2-d_3)(d_1+d_2+d_3).$$

According to (3.10) equality $\alpha_1 = 0$ is equivalent with union of equalities:

$$\alpha_1^{(1)} = d_2 + d_3 - d_1 = 0, \tag{3.11}$$

$$\alpha_1^{(2)} = d_3 + d_1 - d_2 = 0, \tag{3.12}$$

$$\alpha_1^{(3)} = d_1 + d_2 - d_3 = 0. \tag{3.13}$$

Subject to our further consideration is an inequality $\alpha_1^{(1)} \leq 0$.

Lemma 3.3 $\mathbf{1^0}$**.** *For the point* B: $d_2 + d_3 \leq d_1$ *iff* $c \geq a$. $\mathbf{2^0}$**.** *For the point* C: $d_2 + d_3 \leq d_1$ *iff* $b \geq a$.

Remark 3.4 *If* $a > b, c$ *then for points* B *and* C: $\alpha_1 \leq 0$ *and* $\alpha_1^{(1)} > 0$.

Lemma 3.5 *If for point* M: $d_2 + d_3 \leq d_1$, *then we have inequalities*:

$$d_1 + d_2 \geq d_3, \text{ where equality is true for } M = B \text{ and } a = c \tag{3.14}$$

and

$$d_3 + d_1 \geq d_2, \text{ where equality is true for } M = C \text{ and } a = b. \tag{3.15}$$

Proof. It is true

$$(d_1) + d_2 - d_3 \geq (d_2 + d_3) + d_2 - d_3 = 2d_2 \geq 0. \tag{3.16}$$

Hence, the inequality (3.14) follows. Thus, the equality is true only if $M = B$ ($d_2 = 0$) and $a = c$. Analogously, it is true

$$d_3 + (d_1) - d_2 \geq d_3 + (d_2 + d_3) - d_2 = 2d_3 \geq 0. \tag{3.17}$$

Hence, the inequality (3.15) follows. Thus, the equality is true only if $M = C$ ($d_3 = 0$) and $a = b$. ■

Lemma 3.6 $\mathbf{1^0}$**.** *If the point* M *fulfills* $d_2 + d_3 \leq d_1$ *then the following implication is true*:

$$\alpha_1 \leq 0 \Longrightarrow \beta_1 \leq 0. \tag{3.18}$$

$\mathbf{2^0}$**.** *If the point* M *fulfills* $d_3 + d_1 \leq d_2$ *or* $d_1 + d_2 \leq d_3$ *then the following implication is true*:

$$\alpha_1 \leq 0 \Longrightarrow \beta_1 \geq 0. \tag{3.19}$$

Proof. The implications (3.18) and (3.19) have the same assumptions:

$$\begin{aligned} \alpha_1 &= 4d_2^2 d_3^2 - (d_1^2 - d_2^2 - d_3^2)^2 \\ &= \big(2d_2d_3 - d_1^2 + d_2^2 + d_3^2\big)\big(2d_2d_3 + d_1^2 - d_2^2 - d_3^2\big) \leq 0, \end{aligned} \tag{3.20}$$

which follow if the following conjunction is true

$$\big(2d_2d_3 - d_1^2 + d_2^2 + d_3^2\big) \leq 0 \ \text{ and } \ \big(2d_2d_3 + d_1^2 - d_2^2 - d_3^2\big) \geq 0 \tag{3.21}$$

or the conjunction

$$\big(2d_2d_3 - d_1^2 + d_2^2 + d_3^2\big) \geq 0 \ \text{ and } \ \big(2d_2d_3 + d_1^2 - d_2^2 - d_3^2\big) \leq 0. \tag{3.22}$$

1^0. Let $d_2+d_3 \le d_1$ be true. For $M=B$ or $M=C$ implication (3.18) is directly verified. Especially for $M=B$ and $a=c$ or for $M=C$ and $a=b$ equality $\beta_1=0$ is true. Let us assume that $M \ne B, C$ and let us assume that $\alpha_1 \le 0$ in (3.18) be true. On the basis of $d_2+d_3 \le d_1$, according to lemma 3.5 it follows that $d_1+d_2 > d_3$ and $d_3+d_1 > d_2$. Therefore

$$2d_2d_3 - d_1^2 + d_2^2 + d_3^2 = (d_2+d_3)^2 - d_1^2 \le 0 \tag{3.23}$$

and

$$2d_2d_3 + d_1^2 - d_2^3 - d_3^2 = (d_1-d_2+d_3)(d_1+d_2-d_3) > 0. \tag{3.24}$$

From (3.23) and (3.24) we can conclude that the conjunction (3.21) is true and conjunction (3.22) is not true. From the conjunction (3.21) it follows that $d_1^2-d_2^2-d_3^2 \ge 2d_2d_3 > d_2^2+d_3^2-d_1^2$ and from there $d_1^2 > d_2^2+d_3^2$, ie. $\beta_1 < 0$.

2^0. Let $d_3+d_1 \le d_2$ be true. For $M=B$ or $M=C$ implication (3.19) is directly verified. Especially for $M=B$ and $a=c$ or for $M=C$ and $a=b$ equality $\beta_1=0$ is true. Let us assume that $M \ne B, C$ and let us assume that $\alpha_1 \le 0$ in (3.19) be true. On the basis of $d_3+d_1 \le d_2$, according to the lemma analogous to lemma 3.5, it follows $d_2+d_3 > d_1$ and $d_1+d_2 > d_3$. Therefore

$$2d_2d_3 - d_1^2 + d_2^2 + d_3^2 = (d_2+d_3)^2 - d_1^2 > 0 \tag{3.25}$$

and

$$2d_2d_3 + d_1^2 - d_2^3 - d_3^2 = (d_1-d_2+d_3)(d_1+d_2-d_3) \le 0. \tag{3.26}$$

From (3.25) and (3.26) we can conclude that conjunction (3.22) is true and conjunction (3.21) is not true. From conjunction (3.22) follows $d_2^2+d_3^2-d_1^2 \ge 2d_2d_3 > d_1^2-d_2^2-d_3^2$ and therefore, $d_2^2+d_3^2 > d_1^2$, i.e. $\beta_1 > 0$. The implication (3.19) is similarly verified in the case of the inequality $d_1+d_2 \le d_3$. ■

Lemma 3.7 *In the metric space X the condition $d_2+d_3 \le d_1$ is equivalent to the conjunction $\alpha_1 \le 0$ and $\beta_1 \le 0$.*

Proof. ($\Longrightarrow$) Let for the point M the condition $d_2+d_3 \le d_1$ be true. On the basis of equality (3.10) and on the basis of lemma 3.5 it follows $\alpha_1 \le 0$. Therefore, on the basis of lemma 3.6, it follows $\beta_1 \le 0$.

($\Longleftarrow$) Let for the point M conjunction $\alpha_1 \le 0$ and $\beta_1 \le 0$ be true. Then from the conjunction

$$\alpha_1 = (d_2+d_3-d_1)(d_2+d_3+d_1)(2d_2d_3-\beta_1) \le 0 \ \text{ and } \ \beta_1 \le 0 \tag{3.27}$$

follows the condition $d_2+d_3 \le d_1$. ■

Lemma 3.8 *In Ptolemaic metric space X an inequality $\alpha_1^{(1)} \le 0$ is true iff $b \ge a$ or $c \ge a$.*

Proof. On the basis of lemma 3.3 if $a \le c$ then for the point B we have: $\alpha_1^{(1)} = a - c \le 0$ or if $a \le b$ then for the point C we have: $\alpha_1^{(1)} = b - a \le 0$. Conversely, let $a > b, c$ be true. Let $M \in X \backslash \{A, B, C\}$ be any point. Then on the basis of Ptolemaic inequality

$$c \cdot d_3 + b \cdot d_2 \ge a \cdot d_1 \tag{3.28}$$

and assumption $a > b, c$ we can conclude

$$\begin{aligned} (c-a)d_3 + (b-a)d_2 + a(d_2 + d_3 - d_1) \ge 0 \\ \Longrightarrow \alpha_1^{(1)} = d_2 + d_3 - d_1 > 0. \end{aligned} \tag{3.29}$$

By contraposition the statement follows. ■

On the basis of the previous lemmas we can conclude the following theorem is true.

Theorem 3.9 *In the metric space X a point M fulfills $\alpha_1^{(1)} = d_2 + d_3 - d_1 \le 0$ iff $\alpha_1 \le 0$ and $\beta_1 \le 0$ are true. In Ptolemaic metric space X the set of these points M is non-empty iff:*

$$b \ge a \;\; or \;\; c \ge a. \tag{3.30}$$

Inequalities $d_2 + d_3 \le d_1$, $d_3 + d_1 \le d_2$, $d_1 + d_2 \le d_3$

Let us determine set of points M in (Ptolemaic) metric spaces for which some inequalities in (1.3) are true. With respect to point A we formed functions (3.4) and (3.5). Next, with respect to point B let us form functions:

$$\alpha_2 = \alpha_2(M) = 4d_3^2 d_1^2 - \left(d_2^2 - (d_3^2 + d_1^2)\right)^2, \tag{3.31}$$

$$\beta_2 = \beta_2(M) = d_3^2 + d_1^2 - d_2^2 \tag{3.32}$$

and with respect to C point let us form functions:

$$\alpha_3 = \alpha_3(M) = 4d_1^2 d_2^2 - \left(d_3^2 - (d_1^3 + d_2^2)\right)^2, \tag{3.33}$$

$$\beta_3 = \beta_3(M) = d_1^2 + d_2^2 - d_3^2. \tag{3.34}$$

The following equality $\alpha_1 = \alpha_2 = \alpha_3$ is true. Analogously to the theorem 3.9 we can conclude the following theorems are true.

Theorem 3.10 *In the metric space X point M fulfills $\alpha_1^{(2)} = d_3 + d_1 - d_2 \le 0$ iff $\alpha_1 \le 0$ and $\beta_2 \le 0$ are true. In Ptolemaic metric space X the set of these points M is non-empty iff:*

$$c \ge b \;\; or \;\; a \ge b. \tag{3.35}$$

Theorem 3.11 *In the metric space X point M fulfills $\alpha_1^{(3)} = d_1 + d_2 - d_3 \le 0$ iff $\alpha_1 \le 0$ and $\beta_3 \le 0$ are true. In Ptolemaic metric space X the set of these points M is non-empty iff:*

$$a \ge c \;\; or \;\; b \ge c. \tag{3.36}$$

For (Ptolemaic) metric space X the set of the points M with Möbius-Pompeïu metric property fulfill a conjunction:

$$\alpha_1^{(1)} > 0 \quad \text{and} \quad \alpha_1^{(2)} > 0 \quad \text{and} \quad \alpha_1^{(3)} > 0. \tag{3.37}$$

Using theorems 3.9, 3.10 and 3.11 we can determine when some inequalities in (3.37) are not true.

Finally, in the following example let us illustrate a set of points in $\mathbf{R}^2$ with Möbius-Pompeïu metric property, with respect to three fixed points $A, B, C \in \mathbf{R}^2$, if we use metrics $\mathtt{d}$ and $\overline{\mathtt{d}}$ from the example 2.2.

Example 3.12 $\mathbf{1^0}$**.** *Let in the plane* $\mathbf{R}^2$ *the Euclidean metric* $\mathtt{d}$ *is used. By picture* 1 *we illustrate the case of the triangle* ABC *for which* $\mathtt{a} > \mathtt{c} > \mathtt{b}$ *is true. Then* $\alpha_1^{(1)} > 0$ *is true* (*the curve* $\alpha_1^{(1)} = 0$*, on the basis of the theorem* 3.9*, has empty interior and border*)*, otherwise the curves* $\alpha_1^{(2)} = 0$*,* $\alpha_1^{(3)} = 0$ *have non-empty interior and border. We can form a non-degenerative triangle from the remaining points.*

Picture 1.

In the case of the equilateral triangle ABC the curves $\alpha_1^{(1)}\!=\!0$, $\alpha_1^{(2)}\!=\!0$ and $\alpha_1^{(3)}\!=\!0$ transform onto the (smaller) arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ and $\overset{\frown}{AB}$ of the circumcircle. Hence, we have Möbius-Pompeïu theorem in the following form: for equilateral triangle ABC the set of points M in the plane, such that from distances $\mathtt{d}_1 = \mathtt{d}(M,A)$, $\mathtt{d}_2 = \mathtt{d}(M,B)$ and $\mathtt{d}_3 = \mathtt{d}(M,C)$ one can form a degenerative triangle, is circumcircle; from the other points in the plane we can form non-degenerative triangle.

2^0. *Let in the plane $\mathbf{R}^2$ the chordal metric $\overline{\mathtt{d}}$ is used. Let $A,B,C\!\in\!\mathcal{S}\backslash\{(0,0,1)\}$ be points on the unit Riemann sphere $\mathcal{S}$, with uniquely determined projections:*

$$A' = \mathcal{P}^{-1}(A) = a_1 + b_1 i,\; B' = \mathcal{P}^{-1}(B) = a_2 + b_2 i,\; C' = \mathcal{P}^{-1}(C) = a_3 + b_3 i \in \mathbf{C}$$

with inversely stereographical projection from the north pole:

$$\mathcal{P}^{-1} = \mathcal{P}^{-1}(x,y,z) = \Big(\frac{x}{1-z}\Big) + \Big(\frac{y}{1-z}\Big) i : \mathcal{S}\backslash\{(0,0,1)\} \longrightarrow \mathbf{C}.$$

Through points A,B,C on the Riemann sphere let us set great circles (picture 2). In the complex plane we uniquely determine images of great circles as corresponding circles through points A', B', C' (picture 3). By picture 3 we illustrate the case of points A', B', C' for which $\overline{\mathtt{b}} > \overline{\mathtt{c}} > \overline{\mathtt{a}}$ and $\overline{\mathtt{k}} \neq 0$ are true. Then $\overline{\alpha}_1^{(2)}\!>\!0$ (the curve $\overline{\alpha}_1^{(2)}\!=\!0$, on the basis of the theorem 3.10, has empty interior and border), otherwise curves $\overline{\alpha}_1^{(1)}\!=\!0$, $\overline{\alpha}_1^{(3)}\!=\!0$ have non-empty interior and border. From the remaining points we can form a non-degenerative triangle.

Picture 2.

Picture 3.

Let us consider the case when A, B, C are chordally equidistantly arranged points on the Riemann sphere $\mathcal{S}$. Then the set of points M on the Riemann sphere, being such that from chordal distances $\overline{d}_1 = \overline{d}(M, A)$, $\overline{d}_2 = \overline{d}(M, B)$ and $\overline{d}_3 = \overline{d}(M, C)$ one can form a degenerative triangle, is circumcircle; from other points on the Riemann sphere one can form a non-degenerative triangle. Using inverse stereographical projection $\mathcal{P}^{-1}$ we can conclude that analogous statement in complex plane $\mathbf{C}$ is valid if we use chordal metric $\overline{d}$.

REFERENCES

1. D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Chapter XIII: *The Möbius-Neuberg and the Möbius-Pompeïu theorems*, 385-400, Kluwer Academic Publisher 1989.

2. D.S. Mitrinović, J.E. Pečarić, V. Volenec, *History, variations and generalization of the Möbius-Neuberg theorem and the Möbius-Pompeiu theorem*, Bull. Math. Soc. Sci. Math. Roumanie **31** (**79**), No 1, 25-38, 1987.

3. P.A. Hästo, *A New Weighted Metric*: *the Relative Metric* I, J. Math. Anal. Appl. **274**, No 1, 38-58, 2002.